Week 12

Nov 21, 2022

2020A Adv. Cal. II

Let $\vec{r}: D \to S$ be a parametrization of S. The tangent space at $\vec{r}(u,v)$ is spanned by the 2 vectors $\vec{r}_u$ and $\vec{r}_v$. The cross product $\vec{r}_u \times \vec{r}_v$ provides a vector satisfying

$$\vec{r}_u \times \vec{r}_v \cdot \vec{r}_u = 0, \quad \vec{r}_u \times \vec{r}_v \cdot \vec{r}_v = 0,$$

hence $\vec{r}_u \times \vec{r}_v$ lies on the normal direction of S at $\vec{r}(u,v)$.
($\vec{r}_u \times \vec{r}_v \neq \vec{0}$ as $\vec{r}$ is assumed to be regular.)

A choice of a continuous unit normal v.f. $\hat{n}$ is called an orientation of S. A surface with a given unit normal v.f is called an <u>oriented surface</u>. Hence

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}, \quad \text{or} \quad \hat{n} = -\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

Let S be an oriented surface and $\vec{F}$ a continuous v.f. on S, the <u>flux</u> of $\vec{F}$ <u>through</u> S (or <u>across</u> S) is

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma. \quad (1)$$

We have

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_S \vec{F} \cdot (\pm) \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, d\sigma$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA(u,v)$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\pm) \vec{r}_u \times \vec{r}_v \, dA(u,v) \quad (2)$$

($\pm$ means to choose $\hat{n}$ s.t. $= \pm$)

① and ② can be used to find the flux.

e.g. Find the flux of $\vec{F} = (yz, x, -z^2)$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the indicated direction (see Text for the picture).

S is a graph over $[0,1] \times [0,4]$ in the xz-plane.

$$(x, z) \longmapsto (x, x^2, z)$$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2x, -1, 0)$$

We require $\hat{n}$ points to the +ve x direction, so this is the correct choice.

$$\vec{F}(\vec{r}(x,z)) = (x^2 z, x, -z^2)$$

$$\vec{F}(\vec{r}(x,z)) \cdot \vec{r}_x \times \vec{r}_z = (x^2 z, x, -z^2) \cdot (2x, -1, 0)$$

$$= 2x^3 z - x.$$

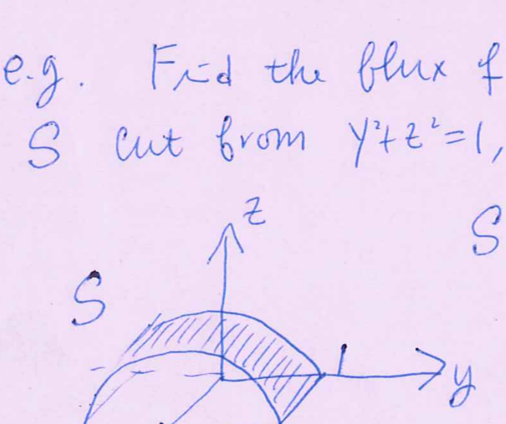
$$\therefore \text{flux} = \iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_{[0,1]\times[0,4]} \vec{F}(\vec{r}(x,z)) \cdot \vec{r}_x \times \vec{r}_z \, dA(x,z) \quad (\text{see ②})$$

$$= \int_0^1 \int_0^4 (2x^3 z - x) \, dz \, dx$$

$$\vdots$$

$$= 2. \#$$

e.g. Find the flux of $\vec{F} = yz\hat{j} + z^2\hat{k}$ outward through the surface S cut from $y^2 + z^2 = 1$, $z \ge 0$, by the plane $x = 0$, $x = 1$.



S is a graph over $[0,1] \times [-1,1]$ in xy-plane

$$(x, y) \longmapsto (x, y, \sqrt{1-y^2})$$

$$\vec{r}_x \times \vec{r}_y = \left(0, \frac{y}{\sqrt{1-y^2}}, 1\right),$$ it is in the right orientation

$$\vec{F}(\vec{r}(x,y)) = y\sqrt{1-y^2}\,\hat{j} + (1-y^2)\hat{k}$$

$$\therefore \text{flux} = \iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_{[0,1]\times[-1,1]} \left(y\sqrt{1-y^2}\,\hat{j} + (1-y^2)\hat{k}\right) \cdot \left(\frac{y}{\sqrt{1-y^2}}\hat{j} + \hat{k}\right) dA(x,y)$$

(by ②)

$$= \int_0^1 \int_{-1}^1 1 \, dy \, dx = 2.$$

(There is No need to use implicit surface as in Text.)

(Cont'd)

Stokes' thm Let S be a piecewise smooth oriented surface whose boundary is a piecewise smooth simple closed curve C. For a smooth v.f. $\vec{F}$ defined in some open region containing S,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, d\sigma .$$

Here $\hat{n}$ is the chosen normal for S, and the orientation of C is anticlockwise w.r.t. $\hat{n}$, and $\nabla \times \vec{F}$ is the curl of $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$$= (P_y - N_z)\hat{i} - (P_x - M_z)\hat{j} + (N_x - M_y)\hat{k}.$$

e.g. Verify Stokes' thm for $x^2+y^2+z^2 = 9, z \geq 0$, outward pointing normal, and its boundary C, $x^2+y^2=9$, for $\vec{F} = y\hat{i} - x\hat{j}$.

Here $\hat{n} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$ (for any sphere at $(0,0,0)$)

$$= \frac{1}{3}(x,y,z)$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = (0,0,-2) = -2\hat{k}$$

$$\nabla \times \vec{F} \cdot \hat{n} = \frac{1}{3}(x,y,z)\cdot(-2\hat{k}) = \frac{-2}{3}z$$

Use $(x,y) \mapsto (x, y, \sqrt{9-x^2-y^2})$, $(x,y) \in D_3$ to parametrize S

$$d\sigma = \sqrt{1+f_x^2+f_y^2}\, dA$$

$$= \sqrt{1+\frac{x^2}{9-x^2-y^2}+\frac{y^2}{9-x^2-y^2}}\, dA$$

$$= \sqrt{\frac{9}{9-x^2-y^2}}\, dA$$

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = \iint_{D_3} -\frac{2}{3}\sqrt{9-x^2-y^2}\,\frac{3}{\sqrt{9-x^2-y^2}}\, dA = -2\iint_{D_3} dA = -18\pi.$$

On the other hand, the correct orientation of C is

$$\vec{r}(\theta) : \theta \mapsto (3\cos\theta, 3\sin\theta, 0), \quad \theta\in[0,2\pi]$$

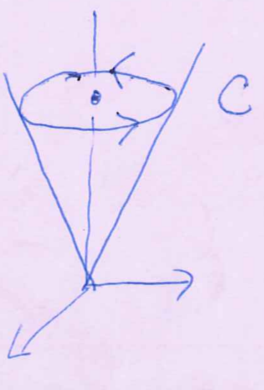
$$\vec{r}'(\theta) = (-3\sin\theta, 3\cos\theta, 0)$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta))\cdot(-3\sin\theta, 3\cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} (3\sin\theta, -3\cos\theta, 0)\cdot(-3\sin\theta, 3\cos\theta, 0)\, d\theta$$

$$= -18\pi. \#$$

e.g. Calculate the circulation around the bounding circle C of the v.f $(x^2-y)\hat{i}+4z\hat{j}+x^2\hat{k}$ where C is the intersection of $z=2$ and $z=\sqrt{x^2+y^2}$, anticlockwise as viewed from above.



Now C is the boundary of 2 surfaces.

S_1, the cone $z=\sqrt{x^2+y^2}$,

S_2, the flat disk.

Apparently, better to work on S_2.

Take the parametrization

$$(x,y) \mapsto (x, y, 2), \quad (x,y) \in D_2$$

for S_2. $\vec{r}_x \times \vec{r}_y = (0, 0, 1)$. The orientation is compatible with the anticlockwise direction of C.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 - y & 4z & x^2 \end{vmatrix} = -4\hat{i} - 2x\hat{j} + \hat{k}$$

$$\iint_{S_2} \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \iint_{D_2} \nabla \times \vec{F} \cdot \vec{r}_x \times \vec{r}_y \, dA$$

$$= \iint_{D_2} (-4\hat{i} - 2x\hat{j} + \hat{k}) \cdot \hat{k}\, dA$$

$$= \iint_{D_2} dA$$

$$= 4\pi .$$

By Stokes', the circulation is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = 4\pi . \quad \#$$

e.g. Let S be the portion of the hyperbolic paraboloid $z = y^2 - x^2$ inside the cylinder $x^2 + y^2 = 1$, with normal pointing upward. Find the circulation of $\vec{F} = y\hat{i} - x\hat{j} + x^2\hat{k}$ around its boundary curve (in anticlockwise viewed from above).

Use the parametrization

$$(x,y) \mapsto (x, y, y^2 - x^2)$$

$$\vec{r}_x \times \vec{r}_y = (-2x, 2y, 1) \quad \text{(the chosen orientation)}$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & x^2 \end{vmatrix} = -2x\hat{j} - 2\hat{k}$$

$$\nabla \times \vec{F} \cdot \vec{r}_x \times \vec{r}_y = (-2x\hat{j} - 2\hat{k}) \cdot (-2x\hat{i} + 2y\hat{j} + \hat{k})$$

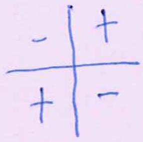
$= -4xy - 2$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \iint_{D_1} (-4xy - 2)\, dA$$

$$= -2 \iint_{D_1} dA$$

$$= -2\pi \quad \#$$

$\left(\iint_{D_1} xy\, dA = 0 \text{ due to symmetry} \right)$



Return to Stokes' theorem.

Some Remarks

(1) Reduction to Green's Theorem.

Regard D as a flat surface S.

$$(x, y) \longleftrightarrow (x, y, 0)$$

Regard $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ as

$$\vec{F}_1 = M(x,y)\hat{i} + N(x,y)\hat{j} + 0\hat{k} \text{ in space.}$$

Choose $\hat{n} = (0,0,1)$ for S_0 and then C_1, the boundary of S_0, will be in anticlockwise direction.

$$C \longleftrightarrow C_1$$

$$(x(t), y(t)) \qquad (x(t), y(t), 0)$$

Stokes' thm applies to S and C_1:

$$\nabla \times \vec{F}_1 = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\hat{k}$$

$$\nabla \times \vec{F}_1 \cdot \hat{n} = (N_x - M_y)\hat{k} \cdot \hat{k} = N_x - M_y$$

$$\therefore \iint_S \nabla\times\vec{F}_1\cdot\hat{n}\,d\sigma = \iint_S (N_x - M_y)\,d\sigma \qquad d\sigma = \sqrt{1+f_x^2+f_y^2}\,dA$$

$$= \sqrt{1+0+0}\,dA$$

$$= \iint_D (N_x - M_y)\,dA \,. \quad ③ \qquad = dA$$

On the other hand,

$$\oint_{C_1} \vec{F}\cdot d\vec{r} = \int_a^b \left(M(x(t),y(t))\,\hat{i} + N(x(t),y(t))\,\hat{j} + 0\,\hat{k}\right)\cdot\left(x'\hat{i}+y'\hat{j}+0\hat{k}\right)dt$$

$$= \int_a^b (Mx' + Ny')\,dt$$

$$= \oint_C M\,dx + N\,dy \,. \quad ④$$

③ + ④

$$\iint_S \nabla\times\vec{F}_1\cdot\hat{n}\,d\sigma = \oint_{C_1}\vec{F}\cdot d\vec{r} \Rightarrow \iint_D (N_x - M_y)\,dA = \oint_C M\,dx + N\,dy \,. \quad \#$$